A Riccati equation in radiative stellar collapse

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Abstract

We model the behaviour of a relativistic spherically symmetric shearing fluid undergoing gravitational collapse with heat flux. It is demonstrated that the governing equation for the gravitational behaviour is a Riccati equation. We show that the Riccati equation admits two classes of new solutions in closed form. We regain particular models, obtained in previous investigations, as special cases. A significant feature of our solutions is the general spatial dependence in the metric functions which allows for a wider study of the physical features of the model, such as the behaviour of the causal temperature in inhomogeneous spacetimes.

1   Introduction

An important application of Einstein's general relativity theory in relativistic astrophysics is the gravitational collapse of a radiating star. Since the first idealized model of a static spherical dust ball proposed by Oppenheimer and Snyder [1], many attempts have been made to describe more realistic situations. The derivation of the junction conditions for a radiating star with the exterior Vaidya metric was first obtained by Santos [2], and this formed the basis on which simple exact radiating models are constructed. The junction conditions were later generalised by Chan *et al* [3] to incorporate pressure anisotropy, and by Maharaj and Govender [4], amongst others, to include the electromagnetic field. Particular solutions of the Einstein equations and boundary conditions have been found to describe the physical scenarios described

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above. These solutions have been used to study the cosmic censorship hypothesis, and various physical features including the adiabatic stability of the model, surface luminosity, and the role of relaxational effects on the thermal evolution in the causal thermodynamic theory.

In the absence of general techniques to analyse the nonlinear coupled partial differential equations in Einstein's theory, many formulations by researchers have sought to simplify the system of equations by considering the special case of shearfree collapse. A successful attempt at an exact model was made by Kolassis *et al* [5], where the assumption of geodesic motion of the fluid particles led to a considerable simplification of the field equations. Herrera *et al* [6] solved Einstein's equations, and reduced the junction condition at the boundary, for shearfree collapse, to a nonlinear ordinary differential equation by requiing that all Weyl tensor components vanish. It was initially thought that this model could only be studied approximately. However this conformally flat model was later solved exactly by Maharaj and Govender [7] by introducing a transformation that linearised the boundary condition. Herrera *et al* [8] subsequently generalised the Maharaj and Govender [7] model and showed that other solutions to the linearised boundary condition existed. In a recent treatment, Misthry *et al* [9] extended the conformally flat model to an exact nonlinear regime by transforming the governing equation to an Abel equation of the first kind. These solutions may be used to study realistic behaviour of the gravitating star with nonlinear boundary conditions.

A natural extension of shearfree models is to include the effects of shear and pressure anisotropy. The origins and effects of anisotropy were first investigated by Herrera and Santos [10] and later by Chan *et al* [11] and Herrera *et al* [12]. The formulation of a shearing model with pressure anisotropy leads to a nonlinear partial differential equation at the boundary; this presents a formidable difficulty in obtaining solutions in closed form. Earlier treatments have been qualitative in nature without exact analytical solutions, and hence researchers have used numerical techniques to study the physical behaviour of the model. Noguiera and Chan [13], assuming separable forms for the metric functions, reduced the boundary condition to a nonlinear ordinary differential equation. They then investigated the equation numerically and a detailed study of the physical features of the model was performed. Recently Naidu *et al* [14] obtained the first exact analytical model with nonzero shear by considering geodesic motion of the fluid particles. Their particular solution, however, has singularities at the stellar core. Maharaj and Misthry [15] obtained two classes of exact solutions, nonsingular at the centre, in which the Naidu *et al* [14] model is contained as a special case. Note that the spatial components of the metric functions have been restricted to particular forms in the Maharaj and Misthry [15] models. This restriction limits the investigation of the relativistic effects such as the relaxation time scales on the model, and consequently it is desirable to obtain a wider class of solutions.

The main objective of this paper is to carry out a systematic study of the governing equation,

at the boundary, for the shearing collapse of a compact radiating stellar fluid model. We seek to obtain a general class of nonsingular exact solutions which allows for flexibility in the choice of physical parameters required to investigate the physical features of the model. It is not desirable to eliminate the inherent nonlinearity at the boundary; instead we seek to transform the governing equation to a familiar form, namely the Riccati equation. In Section 2, we formulate the model using the Einstein field equations together with the junction conditions. In Section 3, we first transform the governing equation into a Riccati equation, and then propose two transformations which lead to separable equations. Two new classes of exact solutions are found. It is shown that solutions found earlier with nonzero shear are contained in our models. In Section 4, we integrate the truncated form of the Maxwell-Cattaneo heat transport equation and obtain an explicit form for the causal temperature. We present profiles for the causal and acausal temperatures and briefly discuss the features of the graphs.

2 Formulation of the model

We seek to model a spherically symmetric star undergoing radiative gravitational collapse with nonzero shear in the context of general relativity. The line element describing the gravitational field for the interior spacetime is taken to be

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where B and Y are functions of both the temporal coordinate t and radial coordinate r. For this model, the fluid four-velocity $u^a = \delta_0^a$ is comoving. The fluid four-acceleration vector $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ, respectively, are given by

$$\dot{u}^a = 0 \tag{2a}$$

$$\Theta = \left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right) \tag{2b}$$

$$\sigma = \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B}\right) \tag{2c}$$

for the line element (1). We observe that the particle trajectories of the collapsing fluid are geodesics because $\dot{u}^a = 0$. However note from (2) that the fluid expansion and the shear may be nonzero in general. A similar analysis was performed by Kolassis *et al* [5] when the shear is vanishing; in that case it was possible to solve the boundary condition and the field equations and obtain an exact solution. The Kolassis *et al* [5] model has a Friedmann limit and the dust cosmological model is regained. It would be interesting to compare the temporal evolution of the model when shear is present. Investigation of the behaviour of the temperature in causal

thermodynamics for geodesic motion has revealed higher central temperatures than the Eckart theory as established by Govender *et al* [16]. In this study we seek to incorporate the effects of shear in the model.

The energy momentum tensor for the interior spacetime has the form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{3}$$

where ρ is the density of the gravitating fluid, p is the isotropic pressure, q_a is the heat flux vector and π_{ab} is the stress tensor. These quantities are measured relative to the four-velocity $\boldsymbol{u}$. The stress tensor can be written explicitly as

$$\pi_{ab} = (p_r - p_t)\left(n_a n_b - \frac{1}{3}h_{ab}\right) \tag{4}$$

where p_r is the radial pressure, p_t is the tangential pressure, and $\boldsymbol{n}$ is the unit radial vector orthogonal to $\boldsymbol{u}$. Hence we have $n^a = \frac{1}{B}\delta_1^a$. The isotropic pressure

$$p = \frac{1}{3}(p_r + 2p_t) \tag{5}$$

relates the radial pressure and the tangential pressure.

It is possible to write the Einstein field equations as the set

$$\rho = 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'}{B}\frac{Y'}{Y}\right) \tag{6a}$$

$$p_r = \left(-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{Y}}{Y}\right) + \frac{1}{B^2}\left(\frac{Y'^2}{Y^2}\right) - \frac{1}{Y^2} \tag{6b}$$

$$p_t = \frac{1}{B^2}\left(-\frac{B'}{B}\frac{Y'}{Y} + \frac{Y''}{Y}\right) - \left(\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y}\right) \tag{6c}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{Y}'}{Y} + \frac{\dot{B}}{B}\frac{Y'}{Y}\right) \tag{6d}$$

for the spacetime (1) and the matter distribution (3). The fluid pressure is anisotropic and the heat flux $q^a = (0, q, 0, 0)$ has only a nonvanishing radial component. We observe that if functional forms for the gravitational potentials B and Y are given then expressions for the matter variables ρ, p_r, p_t and q immediately follow from (6).

The exterior spacetime, describing the region outside the stellar boundary, is described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

4

where $m(v)$ denotes the mass of the fluid as measured by an observer at infinity. The metric (7) describes coherent null radiation which is flowing in the radial direction relative to the hypersurface Σ which represents the boundary of the star. The matching of the exterior spacetime with the interior spacetime leads to the following set of junction conditions on the hypersurface Σ:

$$dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{1/2} dv \tag{8a}$$

$$Y(R_\Sigma, t) = R_\Sigma(v) \tag{8b}$$

$$m(v)_\Sigma = \left[\frac{Y}{2}\left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right)\right]_\Sigma \tag{8c}$$

$$(p_r)_\Sigma = (qB)_\Sigma \tag{8d}$$

The nonvanishing of the radial pressure p_r at Σ leads to an additional differential equation, namely the boundary condition (8d), which has to be satisfied together with the field equations (6). This condition was first established by Santos [2] for shearfree spacetimes, and extended to spacetimes with nonzero shear by Glass [17] and Maharaj and Govender [18], amongst others. In a recent investigation by Di Prisco *et al* [19], the matching conditions applicable to spherically symmetric gravitational collapse with dissipation and nonzero shear have been generalised to include nonadiabatic charged fluids.

3 Solution of the governing equation

The junction condition $(p_r)_\Sigma = (qB)_\Sigma$ becomes

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0 \tag{9}$$

which follows from (6). Equation (9) governs the gravitational behaviour of a radiating star with anisotropic pressure and nonzero shear. To complete the description of the gravitational behaviour of the model we need to integrate the junction condition (9); this will lead to functional forms for the metric functions $B(r,t)$ and $Y(r,t)$. Exact solutions for the junction condition (9) have been extremely difficult to obtain due to the nonlinear nature of the equation. In an earlier study of a shearing radiating model, Noguiera and Chan [13] used numerical techniques to obtain approximate solutions. Ideally an exact solution is desirable in terms of elementary or special functions. An exact solution for this physical model was obtained by Naidu *et al* [14] in terms of the elementary functions. This class of solution is singular at the stellar centre. Maharaj and Misthry [15] extended the Naidu *et al* [14] model and showed that a wider category of solutions are possible; the singularities at the centre were shown to be avoidable.

We seek, in this treatment, to obtain a general class of nonsingular solutions which will allow for an investigation of the physical features of the model. Previous treatments were ad hoc. Our objective is to write (9) in a generic form, and then obtain solutions systematically. It is shown that the model leads to the formation of a Riccati equation where the potential B is the dependent variable. We present, in the following, a method of solving (9) exactly, and find several classes of solutions depending on the form of Y used. We rewrite (9) in the form

$$\dot{B} = \left(\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'} \right) B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y} \tag{10}$$

which is a Riccati equation in the potential B. We demonstrate that two classes of solutions can be found for this Riccati equation.

3.1 The first solution

We seek solutions where Y is a separable function of the form

$$Y = R(r)(t+a)^{2/3} \tag{11}$$

so that the temporal evolution of the model is specified. It is convenient at this point to introduce the transformation

$$B = Z(t+a)^{2/3} \tag{12}$$

Then (10) can be written in the form

$$(t+a)^{2/3}\dot{Z} = \frac{1}{2RR'}(Z^2 - R'^2) \tag{13}$$

Equation (13) is simple and separable with solution

$$Z = R' \left(\frac{1 + g(r)\exp[3(t+a)^{1/3}/R]}{1 - g(r)\exp[3(t+a)^{1/3}/R]} \right) \tag{14}$$

where $g(r)$ is related to an arbitrary constant of integration. Consequently the potential B can be obtained explicitly in the form

$$B = R' \left(\frac{1 + g(r)\exp[3(t+a)^{1/3}/R]}{1 - g(r)\exp[3(t+a)^{1/3}/R]} \right)(t+a)^{2/3} \tag{15}$$

From (11) and (15) we may write the interior metric (1) as

$$ds^2 = -dt^2 + (t+a)^{4/3} \left[R'^2 \left(\frac{1 + g(r)\exp[3(t+a)^{1/3}/R]}{1 - g(r)\exp[3(t+a)^{1/3}/R]} \right)^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{16}$$

which describes the interior spacetime of the radiating star.

The matter variables for the model are given by

$$
\begin{aligned}
\rho &= \frac{4}{3\tilde{t}^2}\left[1 + \frac{3\tilde{t}^{2/3}}{4R^2}\right] - \left[\frac{1 - g\exp[3\tilde{t}^{1/3}/R]}{R\tilde{t}^{2/3}(1 + g\exp[3\tilde{t}^{1/3}/R])}\right]^2 \\
&\quad + \frac{8g\exp[3\tilde{t}^{1/3}/R]}{3\tilde{t}^{5/3}R(1 - g^2\exp 2[3\tilde{t}^{1/3}/R])} \\
&\quad + \frac{4\exp[3\tilde{t}^{1/3}/R](1 - g\exp[3\tilde{t}^{1/3}/R])}{RR'\tilde{t}(1 + g[\exp 3\tilde{t}^{1/3}/R])^3}\left[\frac{g'}{\tilde{t}^{1/3}} - \frac{3g}{R^2}\right] \quad\quad (17a)
\end{aligned}
$$

$$
p_r = -\frac{4g\exp[3\tilde{t}^{1/3}/R]}{\tilde{t}^{4/3}R^2(1 + g\exp[3\tilde{t}^{1/3}/R]^2} \quad\quad (17b)
$$

$$
\begin{aligned}
p_t &= -\frac{2g\exp[3\tilde{t}^{1/3}/R]}{R^2\tilde{t}^{4/3}(1 - g^2\exp 2[3\tilde{t}^{1/3}/R])}\left[1 + \frac{2g\exp[3\tilde{t}^{1/3}/R]}{(1 - g\exp[3\tilde{t}^{1/3}/R])} + \frac{4R}{3\tilde{t}^{1/3}}\right] \\
&\quad - \frac{2\exp[3\tilde{t}^{1/3}/R](1 - g\exp[3\tilde{t}^{1/3}/R])}{RR'\tilde{t}(1 + g\exp[3\tilde{t}^{1/3}/R])^3}\left[\frac{g'}{\tilde{t}^{1/3}} - \frac{3g}{R^2}\right] \quad\quad (17c)
\end{aligned}
$$

$$
q = -\frac{4g\exp[3\tilde{t}^{1/3}/R](1 - g\exp[3\tilde{t}^{1/3}/R])}{R^2R'\tilde{t}^2(1 + g\exp[3\tilde{t}^{1/3}/R])^3} \qu\quad (17d)
$$

which satisfies the Einstein equations (6). For simplicity we have set $\tilde{t} = t + a$.

We have obtained an exact solution to the Einstein field equations (6) that satisfies the boundary condition (9) for a radiating relativistic star. This is a new class of exact solutions where the spatial dependence in the function R is arbitrary. Consequently particular solutions found in the past can be shown to be contained in this class. We observe that when $R = r + b$ the metric (16) becomes

$$
\begin{aligned}
ds^2 &= -dt^2 + (t + a)^{4/3}\left[\left(\frac{1 + g(r)\exp[3(t+a)^{1/3}/(r+b)]}{1 - g(r)\exp[3(t+a)^{1/3}/(r+b)]}\right)^2 dr^2\right. \\
&\quad \left. + (r + b)^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qu\quad (18)
\end{aligned}
$$

Thus we have regained the Maharaj and Misthry [15] model which is regular at the stellar origin. The Naidu *et al* [14] model is regained from (16) when $a = 0$ and $b = 0$. Other choices for the function R are clearly possible: the choice should be such that the model remains regular at the centre and the model is well behaved in the interior. We further observe that the model yields the Friedmann dust model when $g = 0$. In this case we can find coordinates such that

$$
ds^2 = -dt^2 + t^{4/3}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \qu\quad (19)
$$

for which the heat flux vector q_a vanishes and $p_r = p_t = 0$ with $\rho = \frac{4}{3t^2}$ in geodesic motion.

3.2 The second solution

Other solutions to the Riccati equation (10) exist in closed form but these are difficult to find in practice. It is possible to find a second class of solutions to (10) by assuming that

$$Y = R(r)(t + a) \tag{20}$$

In this case we introduce the transformation

$$B = (t + a)Z \tag{21}$$

Then (20) can be written in the form

$$(t + a)\dot{Z} = \frac{R^2 + 1}{2RR'} \left(Z^2 - \frac{R'^2}{R^2 + 1} \right) \tag{22}$$

Equation (22) is a simple equation, separable in the variables Z and t, with solution

$$Z = \frac{R'}{\sqrt{R^2 + 1}} \left(\frac{1 + h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1 - h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}} \right) \tag{23}$$

where $h(r)$ is an arbitrary function of integration. Therefore we can express the metric potential in the form

$$B = \frac{R'}{\sqrt{R^2 + 1}} \left(\frac{1 + h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1 - h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}} \right) (t + a) \tag{24}$$

Then from (20) and (24) we obtain the interior metric

$$ds^2 = -dt^2 + (t + a)^2 \left[\frac{R'^2}{R^2 + 1} \left(\frac{1 + h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}}{1 - h(r)(t + a)^{\sqrt{\frac{R^2+1}{R^2}}}} \right)^2 dr^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{25}$$

which describes the stellar interior.

The Einstein field equations (6) then imply the matter variables which are given by

$$
\rho = \frac{2}{\tilde{t}^2}\left[1+\frac{1}{2R^2}\right]-(3R^2+1)\left[\frac{1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R\tilde{t}\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}\right]^2
$$

$$
+\frac{2h\sqrt{R^2+1}}{RR'}\frac{\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{\tilde{t}^2\left(1-h^2\tilde{t}^{2\sqrt{\frac{R^2+1}{R^2}}}\right)}
$$

$$
+\frac{4\left(R^2+1\right)^2\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R'R\tilde{t}^2\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^2}\left[\frac{h'}{R^2+1}-\frac{h\ln\tilde{t}}{R^9}\right] \tag{26a}
$$

$$
p_r = \frac{1}{\tilde{t}^2}\frac{R^2+1}{R^2}\left[\left(\frac{1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}\right)^2-1\right] \tag{26b}
$$

$$
p_t = -\frac{2h\sqrt{R^2+1}\tilde{t}^{2\sqrt{\frac{R^2+1}{R^2}}}}{R\tilde{t}^2\left(1-h^2\tilde{t}^{2\sqrt{\frac{R^2+1}{R^2}}}\right)}\left[1+\frac{\sqrt{R^2+1}}{RR'}-\frac{1}{R'}\right]-\frac{1}{\tilde{t}^2}
$$

$$
+\frac{1}{\tilde{t}^2}\left(\frac{1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}\right)^2-\frac{4h^2(R^2+1)\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}}{R^2R'\tilde{t}^2\left(1-h^2\tilde{t}^{2\sqrt{\frac{R^2+1}{R^2}}}\right)\left(1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}
$$

$$
-\frac{2(R^2+1)\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\left(1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}{RR'\tilde{t}^2\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^2}\left[h'-\frac{h(R^2+1)\ln\tilde{t}}{R^9\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}\right] \tag{26c}
$$

$$
q = -\frac{4h(R^2+1)^{3/2}\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\left(1-h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)}{R^2R'^2\left(1+h\tilde{t}^{\sqrt{\frac{R^2+1}{R^2}}}\right)^3} \tag{26d}
$$

where we have set $\tilde{t}=t+a$ for convenience.

We have generated another exact solution to the field equations (6) that satisfies the boundary condition (9). Again the spatial dependence in the function is arbitrary and the model is

regular at the centre. We observe that the solution contains an exponential temporal dependence on the spatial function $R(r)$ of the form $t\sqrt{(R^2+1)/R^2}$. Such solutions are difficult to interpret but may be relevant in describing new physical models in the strong gravity regime for gravitational collapse.

4   Causal temperature

The simple forms of the solutions found in this paper, in particular the first solution, permit a study of the physical features. We consider briefly the relativistic effects on the temperature. For a shearing superdense matter distribution, we employ the Maxwell-Cattaneo heat transport equation to investigate the causal thermodynamical behaviour of the model. In the absence of rotation and viscous stress this is given by the truncated version

$$\tau h_a^b \dot{q}_b + q_a = -\kappa (h_a^b \nabla_b T + T \dot{u}_a) \tag{27}$$

where τ is the relaxation time, κ is the thermal conductivity, and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space. When $\tau = 0$ we regain the acausal Fourier heat transport equation. For our model equation (27) may be written as

$$T = -\frac{\tau}{\kappa} \int (qB)^{\cdot} B dr - \frac{1}{\kappa} \int qB^2 dr \tag{28}$$

describing the evolution of the causal temperature. It has been shown in previous investigations of relativistic stellar models that the effect of the relaxation time τ, on the thermal evolution, plays a significant role in the latter stages of collapse [20, 21, 22, 23]. Naidu *et al* [14] showed that in the presence of shear stress, the relaxation time decreases as the collapse proceeds and the central temperature increases. The particular form of the relaxational time τ is dependent on the physical constraints of the model during the latter phases of collapse. We observe that since our solutions have elementary functions with an arbitrary form for the spatial component, it is possible to integrate (28) for different choices of τ. In particular, the effect of decreasing relaxation time with decreasing radius and higher central temperature is possible by incorporating a varying function τ.

In this study we set τ and κ to be constant in order to examine the causal and acausal behaviour in the first solution (18). We need to choose particular forms for the arbitrary function $R(r)$ to complete the integration. As a first example we set $R = r + b$ to obtain for the temperature

$$
\begin{aligned}
T &= \frac{4\tau}{9\kappa\tilde{t}^2} \left(\mathrm{Log}\left[\frac{(1 + g\exp[3\tilde{t}^{1/3}/r + b])^3}{(-1 + g\exp[3\tilde{t}^{1/3}/r + b])^2} \right] \right) \\
&\quad - \frac{4\tau}{3\kappa\tilde{t}^{5/3}} \left(\frac{g\exp[3\tilde{t}^{1/3}/r + b]}{(r + b)(1 + g\exp[3\tilde{t}^{1/3}/r + b])} \right) - \frac{4}{3\kappa\tilde{t}}\mathrm{Tanh}^{-1}(g\exp[3\tilde{t}^{1/3}/r + b]) + f(t) \tag{29}
\end{aligned}
$$

where we have kept g constant, and $f(t)$ is a constant of integration related to luminosity as observed by a distant observer. As a second example we set $R = e^r$ to obtain the temperature

$$
\begin{aligned}
T &= \frac{4\tau}{9\kappa\tilde{t}^2}\left(\mathrm{Log}\left[\frac{(1+g\exp[3\tilde{t}^{1/3}/e^r])^3}{(-1+g\exp[3\tilde{t}^{1/3}/e^r])^2}\right]\right) \\
&\quad -\frac{4\tau}{3\kappa\tilde{t}^{5/3}}\left(\frac{g\exp[3\tilde{t}^{1/3}/e^r]}{e^r(1+g\exp[3\tilde{t}^{1/3}/e^r])}\right) - \frac{4}{3\kappa\tilde{t}}\mathrm{Tanh}^{-1}(g\exp[3\tilde{t}^{1/3}/e^r]) + f(t)
\end{aligned}
\tag{30}
$$

Consequently it is possible to find analytic forms for the causal temperature in terms of elementary functions as shown in (29) and (30). We regain the noncausal (Eckart) temperature profiles when $\tau = 0$. Our simple forms for T assist in studying the evolution of a radiating star in different time intervals. These models provide examples of temperatures where inhomogeneity is directly related to dissipation.

It is possible to qualitatively distinguish the causal and acausal temperatures for the region between the centre and the surface of the star. In Figure 1, we provide plots of the causal (solid line) and Eckart (broken line) temperatures against the radial coordinate on the interval $0 \leq r \leq 1$, where we have selected $\tau = 1$ for simplicity. This figure has been generated with the help of Mathematica. We observe that the temperature is a monotonically decreasing function as we approach the boundary from the stellar centre. Also, it is immediately clear that the causal temperature is everywhere greater than the acausal temperature in the interior of the star. At the boundary Σ, however

$$
T(t, r_\Sigma)_{causal} = T(t, r_\Sigma)_{acausal}
\tag{31}
$$

This simple figure has been generated by assuming a particular constant value for the relaxation time τ and the thermal conductivity κ. Changing the magnitude of these values would produce a change in the separation of the curves but the results do not change qualitatively. For example, in Figure 2, we provide a plot of the causal (solid line) and Eckart (broken line) temperatures for $\tau > 0$. We note that in this case both temperatures decrease more rapidly as we approach the boundary; the value of τ affects the gradient of the temperature. As indicated previously it is possible for the relaxation time τ to vary. The choice for τ should be dictated on physical grounds, e.g. rate of particle production at the stellar surface.

Acknowledgements
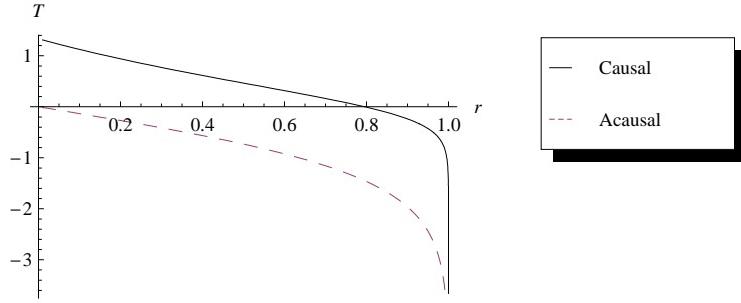
Figure 1: Temperature T vs radial coordinate r ($\tau = 1$)
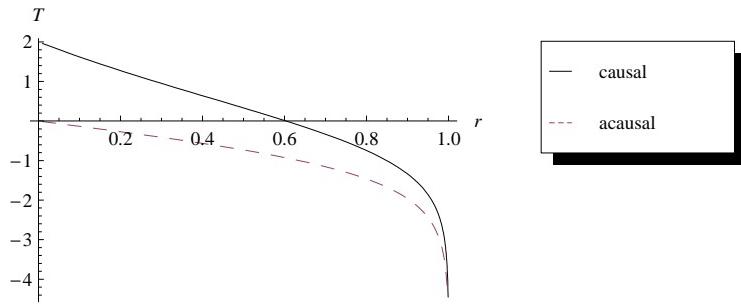
Figure 2: Temperature T vs radial coordinate r ($\tau > 1$)

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